International Journal of Theoretical Physics, Vol. 31, No. 2, 1992

Dichotomies for Linear Impulsive Differential Equations with Variable Structure

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Received October 2, 1990

The notions of *ordinary* and *exponential dichotomy* for linear impulsive differential equations are made precise.

1. INTRODUCTION

Impulsive differential equations (Lakshmikantham and Liu, 1989, and to appear; Lakshmikantham *et al.*, 1989; Milev and Bainov, to appear-*a,b*; Samoilenko and Perestyuk, 1987) are a comparatively new branch of ordinary differential equations. Interest in them has grown recently in relation to the possibility of their application to various branches of science and technology: the theory of automatic control, quantum mechanics, impulse technology, industrial robotics, ecology, and biotechnologies.

In the present paper the ordinary and exponential dichotomies for linear differential equations with variable structure and impulses at fixed moments are investigated.

2. PRELIMINARY NOTES

Let $t_0 < t_1 < \cdots < t_i < \cdots$, $\lim t_i = \infty$ as $i \to \infty$, be a given sequence of real numbers. Consider the linear impulsive differential equation (LIDE) with variable structure and impulses at fixed moments

$$\begin{aligned} \frac{dx}{dt} &= A(t)x, \qquad t \neq t_i \\ x(t_i) &= B_i x(t_i - 0), \qquad i = 1, 2, \ldots \end{aligned} \tag{1}$$

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0020-7748/92/0200-0353$06.50/0

where the $n \times n$ coefficient matrix $A(t)$ is piecewise continuous in the interval $[t_0, +\infty)$ with points of discontinuity of the first kind at $t = t_i$, $i = 1, 2, \ldots,$ and the impulse matrices B_i, $i = 1, 2, \ldots,$ are constant. The underlying vector space E is $\mathbb{R}^n$ or $\mathbb{C}^n$.

Remark 1. When the coefficient matrix $A(t)$ is continuous on the interval $[t_0, +\infty)$, then the LIDE (1) just has an impulse effect but not a variable structure.

The solutions $x(t)$ defined in the interval $[t_k, +\infty)$ are continuously differentiable for $t \neq t_i$ with points of discontinuity of the first kind at $t = t_i$, $i > k$. Note that $x(t_i) := x(t_i + 0)$, $i = 1, 2, \ldots$.

The fundamental matrix $X(t, s)$ of the LIDE (1) for $t \geq s$, $t \in [t_m, t_{m+1} - 0]$, $s \in [t_{j-1}, t_j - 0]$, $m \geq j - 1$, admits the representation

$$X(t, s) = U(t)U^{-1}(t_m)B_m U(t_m - 0) \cdots U^{-1}(t_j)B_j U(t_j - 0)U^{-1}(s) \qquad (2)$$

where $U(t)$ is the fundamental matrix of the equation $dx/dt = A(t)x$. The fundamental matrix $X(t, s)$ is invertible if and only if the impulse matrices B_i, $j \leq i \leq m$, are nonsingular.

3. MAIN RESULTS

Denote by L_k, $k = 0, 1, 2, \ldots,$ the linear space of solutions $x(t)$ of the LIDE (1) defined in the interval $[t_k, +\infty)$. Let $e_j = \text{col}(\delta_1^j, \ldots, \delta_n^j)$, where $\delta_i^j = 0$ for $i \neq j$; $\delta_i^i = 1$ is Kronecker's symbol; and $\text{col}(\cdots)$ stands for a column vector. The solutions $x_j(t) = X(t, t_k + 0)e_j$, $j = 1, 2, \ldots, n$, are linearly independent as elements of the linear space L_k. Their restrictions to the interval $[t_{k+1}, +\infty)$ as elements of the linear space L_{k+1} are linearly dependent if and only if the impulse matrix B_{k+1} is singular. It is only in this case that both the merging of solutions at the point t_{k+1} and the noncontinuability to the left of some solutions of L_{k+1} are observed. Each solution $x(t)$ of L_k with initial value $x(t_k) = \text{col}(\lambda_1, \ldots, \lambda_n)$ is a linear combination of the solutions $x_j(t)$, $j = 1, 2, \ldots, n$:

$$x(t) = X(t, t_k)x(t_k) = \lambda_1 x_1(t) + \cdots + \lambda_n x_n(t)$$

i.e., L_k, $k = 0, 1, 2, \ldots,$ are n-dimensional linear spaces.

Due to the presence for linear impulsive differential equations of phenomena such as merging of the solutions and noncontinuability to the left, it is appropriate to introduce new notions for ordinary and exponential dichotomies so as to take into account the specific character of this class of ordinary differential equations.

Definition 1. The LIDE (1) is said to have *exponential dichotomies* if there exist positive constants α, β, and N and if for any nonnegative integer k the following conditions hold:

D1. The underlying vector space E is split up into a direct sum of $\mathbb{R}$- (or $\mathbb{C}$-) linear subspaces $E = Y_k \oplus Z_k$.

D2. All solutions x, y, and z of the LIDE (1) for which $x = y + z$, $y(t_k) \in Y_k$, and $z(t_k) \in Z_k$ satisfy the conditions

$$|y(t)| \le N e^{-\alpha(t-s)}|x(s)| \qquad \text{for} \quad t \ge s \ge t_k$$
$$|z(t)| \le N e^{-\beta(s-t)}|x(s)| \qquad \text{for} \quad s \ge t \ge t_k$$

Definition 2. The LIDE (1) is said to have an *ordinary dichotomy* if there exists a positive constant N and if for any nonnegative integer k conditions (D1) and (D3) hold:

D3. All solutions x, y, and z of the LIDE (1) for which $x = y + z$, $y(t_k) \in Y_k$, and $z(t_k) \in Z_k$ satisfy the conditions

$$|y(t)| \le N|x(s)| \qquad \text{for} \quad t \ge s \ge t_k$$
$$|z(t)| \le N|x(s)| \qquad \text{for} \quad s \ge t \ge t_k$$

Definition 3. The LIDE (1) is said to have a *weak exponential dichotomy* (*weak ordinary dichotomy*) with respect to the space of solutions L_k if conditions (D1) and (D2) [(D1) and (D3)] hold only for the solutions of the space L_k, where k is a fixed number.

Remark 2. Condition (D2) can be replaced by the equivalent condition (D20):

D20. All solutions x, y, and z of the LIDE (1) for which $x = y + z$, $y(t_k) \in Y_k$, and $z(t_k) \in Z_k$ satisfy the conditions

$$|y(t)| \le N_1 e^{-\alpha(t-s)}|y(s)| \qquad \text{for} \quad t \ge s \ge t_k$$
$$|z(t)| \le N_1 e^{-\beta(s-t)}|z(s)| \qquad \text{for} \quad s \ge t \ge t_k$$
$$|y(t)| \le N_1|x(t)| \qquad \text{for} \quad t \ge t_k$$

Remark 3. Condition (D2) can be also replaced by the equivalent condition (D21):

D21. There exist projectors P_k ($P_k^2 = P_k$) with ranges $R(P_k) = Y_k$ and nullspaces $\operatorname{Ker} P_k = Z_k$ such that

$$|X(t, t_k)P_k\eta| \le N e^{-\alpha(t-s)}|X(s, t_k)\eta| \qquad \text{for} \quad t \ge s \ge t_k$$
$$|X(t, t_k)(I - P_k)\eta| \le N e^{-\beta(s-t)}|X(s, t_k)\eta| \qquad \text{for} \quad s \ge t \ge t_k$$

where η is an arbitrary vector.

Note that $X(t_k, t_k) = I$ (the unit matrix).

Remark 4. For the projectors P_k and P_m, $k \geq m$, the equality $P_k X(t_k, t_m) = X(t_k, t_m) P_m$ is valid.

Remark 5. Analogously, condition (D3) of Definition 2 can be replaced by the equivalent conditions (D30) or (D31), which are obtained respectively from conditions (D20) or (D21) for $\alpha = \beta = 0$.

Proposition 1. If the LIDE (1) has a weak exponential dichotomy (weak ordinary dichotomy) with respect to the space L_k, then:

1.1. (Coppel, 1978, pp. 16, 17). For any projector P with range $R(P) = Y_k$ there exists a constant $N = N(P)$ such that condition (D21) [(D31)] hold.

1.2. For any subspace Z supplementary to Y_k there exists a constant $N = N(Z)$ so that condition (D2) [(D3)] hold.

Assertion 1.2 follows from assertion 1.1, choosing a projector P for which $R(P) = Y_k$ and Ker $P = Z$.

Proposition 2. If the LIDE (1) has a weak exponential dichotomy with respect to the space L_k, then the subspace Y_k is uniquely determined and consists of the initial values $y(t_k)$ of all bounded solutions of L_k.

Proof. By Definition 3 for the solution $y(t)$ with initial value $y(t_k) \in Y_k$, condition (D20) is valid:

$$|y(t)| \leq N_1 \, e^{-\alpha(t - t_k)} |y(t_k)| \leq N_2 \, e^{-\alpha t}$$

where the constant $N_2 = N_1 \, e^{\alpha t_k} |y(t_k)|$, i.e., the solution $y(t)$ with initial value $y(t_k) \in Y$, exponentially tends to zero as $t \to \infty$. Let $z(t)$ be a solution with initial value $z(t_k) \notin Y_k$ and let Z be the subspace through $z(t_k)$ supplementary to Y_k. By assertion 1.2 and Remark 2 for the solution $z(t)$, condition (D20) is valid:

$$|z(t_k)| \leq N_1 \, e^{-\beta(s - t_k)} |z(s)|$$

i.e., $|z(s)| \geq N_3 \, e^{\beta s}$, where the constant $N_3 = N_1^{-1} \, e^{\beta t_k} |z(t_k)|$. Hence the solution $z(s)$ with initial value $z(t_k) \notin Y$ exponentially tends to infinity as $s \to \infty$. ■

Proposition 3 (Coppel, 1978, 2, Pr. 2, p. 17). Let the LIDE (1) have a weak ordinary dichotomy with respect to the space L_k and let Y'_k be the subspace formed by the initial values of all solutions of L_k tending to zero as $t \to \infty$. Let Y''_k be the subspace formed by the initial values of all bounded solutions of L_k. Then $Y'_k \subset Y_k \subset Y''_k$ and any other subspace $\tilde{Y}_k$ for which $Y'_k \subset \tilde{Y}_k \subset Y''_k$ also induces a weak ordinary dichotomy with respect to L_k.

Definition 4 (Milev and Bainov, to appear-*b*). The LIDE (1) is said to be *weakly uniformly exponentially stable* with respect to the space of solutions

L_k (k is a fixed number) if there exist positive constants α and N such that for any solution $x \in L_k$ the following inequality is valid:

$$|x(t)| \leq N e^{-\alpha(t-s)} |x(s)| \qquad \text{for} \quad t \geq s \geq t_k$$

Definition 5 (Milev and Bainov, to appear-*a*). If the constant α in Definition 4 equals zero, then the LIDE (1) is said to be *weakly uniformly stable* with respect to the space L_k.

Denote by $\tilde{B}_k$ the map

$$\tilde{B}_k(\eta) = B_k U(t_k - 0) U^{-1}(t_{k-1}) \eta: \quad E \to E$$

Proposition 4. If the LIDE (1) has a weak exponential dichotomy with respect to L_k induced by the subspace Y_k, then the subspace

$$Y_{k-1} = \tilde{B}_k^{-1}(Y_k) = \{\eta \mid \tilde{B}_k(\eta) \in Y_k\}$$

also induces a weak exponential dichotomy (possibly degenerate) with respect to L_{k-1}:

(a) If the range $R(\tilde{B}_k) \subset Y_k$, then the LIDE (1) is weakly uniformly exponentially stable with respect to L_{k-1} (a degenerate dichotomy).

(b) If $R(\tilde{B}_k) \cap Y_k = \{0\}$, then each solution $y \in L_{k-1}$ with initial value $y(t_{k-1}) \in Y_{k-1}$ is identically zero in L_k and each solution $z \in L_{k-1}$ with initial value $z(t_{k-1}) \notin Y_{k-1}$ tends exponentially to infinity (a degenerate dichotomy).

(c) If $R(\tilde{B}_k) \not\subset Y_k$ and $R(\tilde{B}_k) \cap Y_k \neq \{0\}$, then Y_{k-1} induces a weak exponential dichotomy with respect to L_{k-1}.

Proof. By the lemma of Gronwall–Bellman for any τ_1, $\tau_2 \in [t_{k-1}, t_k - 0]$ the following inequality is valid:

$$|U(\tau_1) U^{-1}(\tau_2)| \leq \exp \int_{t_{k-1}}^{t_k} |A(\theta)| \, d\theta = a_k$$

In the cases (b) and (c) let the space Z_{k-1} be supplementary to Y_{k-1} and in the case (a) $Z_{k-1} = \varnothing$ and let Z_k be a supplementary subspace to Y_k which contains the subspace $\tilde{B}_k(Z_{k-1})$. The LIDE (1) has a weak exponential dichotomy with respect to L_k and by assertion 1.2 condition (D20) is valid. Consider the solutions x, y, and z for which $x = y + z$, $y(t_{k-1}) \in Y_{k-1}$, and $z(t_{k-1}) \in Z_{k-1}$. Their restrictions to the interval $[t_k, +\infty)$ belong to L_k, $y(t_k) \in Y_k$, and $z(t_k) \in Z_k$. Hence

$$|y(t)| \leq N e^{-\alpha(t-s)} |y(s)| \qquad \text{for} \quad t \geq s \geq t_k$$

$$|z(t)| \leq N e^{-\beta(s-t)} |z(s)| \qquad \text{for} \quad s \geq t \geq t_k$$

$$|z(t)| \leq N |x(t)| \qquad \text{for} \quad t \geq t_k$$

If $t_{k-1} \leq s < t_k \leq t$, then

$$
\begin{aligned}
|y(t)| &\leq N\, e^{-\alpha(t-t_k)}|y(t_k)| \\
&= N\, e^{-\alpha(t-s)}\, e^{\alpha(t_k-s)}|B_k U(t_k-0)\, U^{-1}(s) y(s)| \\
&\leq N|B_k| a_k\, e^{\alpha(t_k-t_{k-1})}\, e^{-\alpha(t-s)}|y(s)|
\end{aligned}
$$

If $t_{k-1} \leq s \leq t < t_k$, then

$$
|y(t)| = |U(t)\, U^{-1}(s) y(s)| \leq a_k\, e^{\alpha(t_k-t_{k-1})}\, e^{-\alpha(t-s)}|y(s)|
$$

Hence for any $t \geq s \geq t_{k-1}$ the inequality

$$
|y(t)| \leq N_1\, e^{-\alpha(t-s)}|y(s)|
$$

is valid, where

$$
N_1 = \max(N,\, N|B_k| a_k\, e^{\alpha(t_k-t_{k-1})},\, a_k\, e^{\alpha(t_k-t_{k-1})})
$$

The map $\tilde{B}_k\colon Z_{k-1} \to \tilde{B}_k(Z_{k-1})$ is a bijection, since

$$
Z_{k-1} \cap \tilde{B}_k^{-1}(0) \subset Z_{k-1} \cap Y_{k-1} = \{0\}
$$

Hence $|z(t_{k-1})| \leq |\tilde{B}_k^{-1}|\, |z(t_k)|$. Then, for $t_{k-1} \leq t < t_k$,

$$
\begin{aligned}
|z(t)| &= |U(t)\, U^{-1}(t_{k-1}) z(t_{k-1})| \\
&\leq a_k |\tilde{B}_k^{-1}|\, |z(t_k)| \\
&\leq a_k |\tilde{B}_k^{-1}| N |x(t_k)| \\
&= a_k |\tilde{B}_k^{-1}| N |B_k U(t_k-0)\, U^{-1}(t) x(t)| \\
&\leq a_k^2 |\tilde{B}_k^{-1}| N |B_k|\, |x(t)|
\end{aligned}
$$

i.e., $|z(t)| \leq N_2 |x(t)|$, where $N_2 = a_k^2 |\tilde{B}_k^{-1}| N |B_k|$.

If $t_{k-1} \leq t < t_k \leq s$, then

$$
\begin{aligned}
|z(t)| &\leq a_k |\tilde{B}_k^{-1}|\, |z(t_k)| \\
&\leq a_k |\tilde{B}_k^{-1}| N\, e^{-\beta(s-t_k)}|z(s)| \\
&\leq a_k |\tilde{B}_k^{-1}| N\, e^{-\beta(s-t)}\, e^{\beta(t_k-t)}|z(s)| \\
&\leq a_k |\tilde{B}_k^{-1}| N\, e^{\beta(t_k-t_{k-1})}\, e^{-\beta(s-t)}|z(s)|
\end{aligned}
$$

If $t_{k-1} \leq t \leq s < t_k$, then

$$
|z(t)| = |U(t)\, U^{-1}(s) z(s)| \leq a_k\, e^{\beta(t_k-t_{k-1})}\, e^{-\beta(s-t)}|z(s)|
$$

Hence for any $s \geq t \geq t_{k-1}$ the inequality

$$
|z(t)| \leq N_3\, e^{-\beta(s-t)}|z(s)|
$$

is valid, where

$$N_3 = \max(N, a_k|\tilde{B}_k^{-1}|N\, e^{\beta(t_k - t_{k-1})}, a_k\, e^{\beta(t_k - t_{k-1})})$$

Choosing $\tilde{N} = \max(N_1, N_2, N_3)$, we obtain Proposition 4. ■

For $\alpha = \beta = 0$ we obtain the following assertion.

Proposition 5. Let the LIDE (1) have a weak ordinary dichotomy with respect to L_k and let Y_k' and Y_k'' be the subspaces defined in the condition of Proposition 3.

(a) If $R(\tilde{B}_k) \subset Y_k''$, then the LIDE (1) is weakly uniformly stable with respect to L_{k-1}.

(b) If $R(\tilde{B}_k) \cap Y_k'' = \{0\}$, then each solution $y \in L_{k-1}$ with initial value $y(t_{k-1}) \in \tilde{B}_k^{-1}(Y_k'')$ is identically zero in L_k and each solution $z \in L_{k-1}$ with initial value $z(t_{k-1}) \notin \tilde{B}_k^{-1}(Y_k'')$ tends to infinity.

(c) If $R(\tilde{B}_k) \not\subset Y_k''$ and $R(\tilde{B}_k) \cap Y_k'' \neq \{0\}$, then $\tilde{B}_k^{-1}(Y_k'')$ induces a weak ordinary dichotomy with respect to L_{k-1} and if $R(\tilde{B}_k) \cap Y_k' \neq \{0\}$ as well, then each subspace Y_{k-1} for which $\tilde{B}_k^{-1}(Y_k') \subset Y_{k-1} \subset \tilde{B}_k^{-1}(Y_k'')$ induces a weak ordinary dichotomy with respect to L_{k-1}.

Corollary 1. Let the impulse matrix B_k of the LIDE (1) be nondegenerate. If the equation has a weak exponential dichotomy or a weak ordinary dichotomy with respect to the space L_k, then it has a weak exponential dichotomy or respectively weak ordinary dichotomy (nondegenerate) with respect to the space L_{k-1} as well.

Proposition 6. Let the LIDE (1) have a weak exponential dichotomy (weak ordinary dichotomy) with respect to the space L_{k-1} and let the impulse matrix B_k be nonsingular. Then the equation has a weak exponential dichotomy (weak ordinary dichotomy) with respect to the space L_k as well.

Proof. The assertion follows from the fact that each solution of L_k is a restriction of a solution of L_{k-1}, since the impulse matrix B_k is nonsingular. ■

Proposition 7. Let the impulse matrices B_i, $i = 1, 2, \ldots,$ of the LIDE (1) be nonsingular. If the equation has a weak exponential dichotomy (weak ordinary dichotomy) with respect to some fixed space L_k, then the equation has an exponential dichotomy (ordinary dichotomy), too.

Proof. Proposition 7 follows from Corollary 1 and Proposition 6. ■

Remark 6. When the impulse matrix B_k is singular, then it is possible for the LIDE (1) to have a weak exponential dichotomy (weak ordinary dichotomy) with respect to the space L_{k-1} and to have no weak exponential dichotomy (weak ordinary dichotomy) with respect to the space L_k. We shall illustrate this by the following example.

Example 1. Consider the LIDE (1), where the impulses are at the moments $t_i=i$, $i=0, 1, 2, \ldots$, and the coefficient and impulse matrices are

$$A(t)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad B_i=\begin{cases} B & \text{for} \quad i=1 \\ I & \text{for} \quad i\geq 2 \end{cases} \qquad B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A straightforward verification yields that the equation has a weak exponential dichotomy with respect to the space L_0, since the impulse at the moment $t_1=1$ crumples the "inconvenient" solutions. The equation has neither a weak exponential dichotomy nor a weak ordinary dichotomy with respect to the spaces L_i, $i\geq 1$, since there the problem coincides with the classical one and the eigenvalues of the matrix $A(t)$ with zero real part are not semisimple.

Remark 7. If in Example 1 we define the impulse matrices by the equality

$$B_i=\begin{cases} B & \text{for} \quad i=j^2 \\ I & \text{for} \quad i\neq j^2, \quad j=1, 2, \ldots \end{cases}$$

then a straightforward verification yields that the new equation will have a weak exponential dichotomy with respect to each of the spaces L_k but will have neither an exponential dichotomy nor an ordinary dichotomy. But if in Example 1 we define the impulse matrices by the equality

$$B_i=\begin{cases} B & \text{for} \quad i=10j \\ I & \text{for} \quad i\neq 10j, \quad j=1, 2, \ldots \end{cases}$$

then the new equation will have an exponential dichotomy with constants $\alpha=\beta=1$ and $N=\exp 13$.

ACKNOWLEDGMENT

The present investigation was supported by the Ministry of Culture, Science and Education of the People's Republic of Bulgaria under Grant 61.

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